

CHAINS OF SEMIDUALIZING MODULES

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ABSTRACT. Let (R, \mathfrak{m}, k) be a commutative Noetherian local ring. We study the suitable chains of semidualizing R -modules. We prove that when R is Artinian, the existence of a suitable chain of semidualizing modules of length $n = \max \{ i \geq 0 \mid \mathfrak{m}^i \neq 0 \}$ implies that the Poincaré series of k and the Bass series of R have very specific forms. Also, in this case we show that the Bass numbers of R are strictly increasing. This gives an insight into the question of Huneke about the Bass numbers of R .

1. INTRODUCTION

In this paper R is a commutative Noetherian local ring with maximal ideal \mathfrak{m} and residue field k . Foxby [6], Vasconcelos [19] and Golod [9] independently initiated the study of semidualizing modules. A finite (i.e. finitely generated) R -module C is called *semidualizing* if the natural homothety map $\chi_C^R : R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism and $\text{Ext}_R^{\geq 1}(C, C) = 0$ (see [11, Definition 1.1]). Trivial examples of semidualizing R -modules are R itself and a dualizing R -module when one exists. The set of all isomorphism classes of semidualizing R -modules is denoted by $\mathfrak{S}_0(R)$, and the isomorphism class of a semidualizing R -module C is denoted $[C]$. The set $\mathfrak{S}_0(R)$ has a rich structure, for instance, it comes equipped with an ordering based on the notion total reflexivity. For semidualizing R -modules B and C , we write $[C] \trianglelefteq [B]$ whenever B is totally C -reflexive. In [8], Gerko defines chains in $\mathfrak{S}_0(R)$. A *chain* in $\mathfrak{S}_0(R)$ is a sequence $[C_n] \trianglelefteq \cdots \trianglelefteq [C_1] \trianglelefteq [C_0]$, and such a chain has length n if $[C_i] \neq [C_j]$ whenever $i \neq j$. In [16], Sather-Wagstaff uses the length of chains in $\mathfrak{S}_0(R)$ to provide a lower bound for the cardinality of $\mathfrak{S}_0(R)$.

The well-known result of Foxby [6], Reiten [14] and Sharp [18] is that R admits a dualizing module if and only if R is Cohen-Macaulay and a homomorphic image of a local Gorenstein ring. Then Jorgensen et. al. [13], characterize the Cohen-Macaulay local rings which admit dualizing modules and non-trivial semidualizing modules. Recently, Amanzadeh and Dibaei [1], characterize the Cohen-Macaulay local rings which admit dualizing modules and suitable chains of semidualizing modules.

In [8], Gerko has shown that, when R is Artinian, the existence of collections of strongly Tor-independent semidualizing R -modules (this condition conjecturally equivalent to the existence of a corresponding long chain of semidualizing modules) implies that the Poincaré

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series of k has a very specific form. We show that the [8, Conjecture 4.6] holds true for suitable chains in $\mathfrak{G}_0(R)$ (see Theorem 3.5). This result implies that the set $\mathfrak{G}_0(R)$ can not contain suitable chains of arbitrary length. Indeed, the length of a suitable chain in $\mathfrak{G}_0(R)$ is less than the number $l = \min \{ i > 0 \mid \mathfrak{m}^i = 0 \}$.

There are several works which were motivated by the following questions of Huneke about the Bass numbers $\mu_R^i(R) = \text{rank}_k(\text{Ext}_R^i(k, R))$. For instance, see [3], [5], [12] and [15]. However, each of the following questions is still open in general.

Question. *Let R be a Cohen-Macaulay ring.*

- (a) *If the sequence $\{\mu_R^i(R)\}$ is bounded above by a polynomial in i , must R be Gorenstein?*
- (b) *If R is not Gorenstein, must the sequence $\{\mu_R^i(R)\}$ grow exponentially?*

In [15], Sather-Wagstaff has proved that if there exists a chain of semidualizing R -modules of length n , then the Bass series of R is a product of n power series with positive integer coefficients, and the Bass numbers of R are bounded below by a polynomial of degree $n - 1$. In Theorem 3.8, we prove that an Artinian ring R with $\mathfrak{m}^{n+1} = 0$ is $SD(n)$ -full if and only if there is a suitable chain in $\mathfrak{G}_0(R)$ of length n . Then by this result and some results of [8], we show that the existence of a suitable chain in $\mathfrak{G}_0(R)$ implies that the Bass series of R has very specific form and the sequence $\{\mu_R^i(R)\}$ is strictly increasing (see Proposition 3.11 and Remark 3.12).

2. PRELIMINARIES

This section contains definitions and background material.

Definition 2.1. ([11, Definition 2.7] and [17, Theorem 5.2.3 and Definition 6.1.2]) Let C be a semidualizing R -module. A finite R -module M is *totally C -reflexive* when it satisfies the following conditions:

- (i) the natural homomorphism $\delta_M^C : M \longrightarrow \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism, and
- (ii) $\text{Ext}_R^{\geq 1}(M, C) = 0 = \text{Ext}_R^{\geq 1}(\text{Hom}_R(M, C), C)$.

Fact 2.2. Define the order \leq on $\mathfrak{G}_0(R)$. For $[B], [C] \in \mathfrak{G}_0(R)$, write $[C] \leq [B]$ when B is totally C -reflexive (see, e.g., [16]). This relation is reflexive and antisymmetric [7, Lemma 3.2], but it is not known whether it is transitive in general. Also, write $[C] < [B]$ when $[C] \leq [B]$ and $[C] \neq [B]$.

In the case D is a dualizing R -module, one has $[D] \leq [B]$ for any semidualizing R -module B , by [10, (V.2.1)].

If $[C] \leq [B]$ then $\text{Hom}_R(B, C)$ is a semidualizing and $[C] \leq [\text{Hom}_R(B, C)]$ ([4, Theorem 2.11]).

The following notations are taken from [16].

Notation 2.3. Assume that $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a chain in $\mathfrak{G}_0(R)$. For each $i \in [n] = \{1, \dots, n\}$ set $B_i = \text{Hom}_R(C_{i-1}, C_i)$. For each sequence of integers $\mathbf{i} = \{i_1, \dots, i_j\}$ with $j \geq 1$ and $1 \leq i_1 < \cdots < i_j \leq n$, set $B_{\mathbf{i}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}$. ($B_{\{i_1\}} = B_{i_1}$ and set $B_{\emptyset} = C_0$).

Fact 2.4. If $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a chain in $\mathfrak{G}_0(R)$ such that $C_0 \cong R$, then by [8, Corollary 3.3], for each $i \in [n]$ we have the following isomorphisms

$$C_i \cong C_0 \otimes_R \text{Hom}_R(C_0, C_1) \otimes_R \cdots \otimes_R \text{Hom}_R(C_{i-1}, C_i) \cong B_1 \otimes_R \cdots \otimes_R B_i.$$

For a semidualizing R -module C , set $(-)^{\dagger C} = \text{Hom}_R(-, C)$.

Definition 2.5. ([16, Remark 3.4] and [1, Definition 3.1]) Let $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ be a chain in $\mathfrak{G}_0(R)$ of length n . For each sequence of integers $\mathbf{i} = \{i_1, \dots, i_j\}$ such that $j \geq 0$ and $1 \leq i_1 < \cdots < i_j \leq n$, set $C_{\mathbf{i}} = C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \cdots \dagger_{C_{i_j}}}$. (When $j = 0$, set $C_{\mathbf{i}} = C_{\emptyset} = C_0$).

We say that the above chain is *suitable* if $C_0 = R$ and $C_{\mathbf{i}}$ is totally C_t -reflexive, for all \mathbf{i} and t with $i_j \leq t \leq n$.

Note that if $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [R]$ is a suitable chain, then $C_{\mathbf{i}}$ is a semidualizing R -module for each $\mathbf{i} \subseteq [n]$.

Proposition 2.6. Assume that $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$.

- (1) ([16, Lemma 4.5] and [1, Remark 3.3]) For each sequence $\mathbf{i} = \{i_1, \dots, i_j\} \subseteq [n]$, the R -module $B_{\mathbf{i}}$ is a semidualizing.
- (2) ([16, Lemma 4.6] and [1, Remark 3.3]) If $\mathbf{i}, \mathbf{s} \subseteq [n]$ are two sequences with $\mathbf{s} \subseteq \mathbf{i}$, then $[B_{\mathbf{i}}] \trianglelefteq [B_{\mathbf{s}}]$ and $\text{Hom}_R(B_{\mathbf{s}}, B_{\mathbf{i}}) \cong B_{\mathbf{i} \setminus \mathbf{s}}$.
- (3) ([16, Theorem 3.3] and [1, Remark 3.3]) $|\mathfrak{G}_0(R)| \geq |\{[C_{\mathbf{i}}] \mid \mathbf{i} \subseteq [n]\}| = 2^n$.
- (4) ([16, Theorem 4.7] and [1, Remark 3.3]) $\{[B_{\mathbf{u}}] \mid \mathbf{u} \subseteq [n]\} = \{[C_{\mathbf{i}}] \mid \mathbf{i} \subseteq [n]\}$.

For an R -module M , the i th Bass number of M is the integer $\mu_R^i(M) = \text{rank}_k(\text{Ext}_R^i(k, M))$, and the Bass series of M is the formal Laurent series $I_R^M(t) = \sum_{i \in \mathbb{Z}} \mu_R^i(M) t^i$. The i th Betti number of M is the integer $\beta_i^R(M) = \text{rank}_k(\text{Ext}_R^i(M, k)) = \text{rank}_k(\text{Tor}_i^R(k, M))$, and the Poincaré series of M is the formal Laurent series $P_M^R(t) = \sum_{i \in \mathbb{Z}} \beta_i^R(M) t^i$.

3. RESULTS

First we investigate suitable chains modulo regular sequences.

Proposition 3.1. Assume that $\mathbf{x} = x_1, \dots, x_d$ is an R -regular sequence and $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n . Then $[\overline{C}_n] \triangleleft \cdots \triangleleft [\overline{C}_1] \triangleleft [\overline{C}_0]$ is also a suitable chain in $\mathfrak{G}_0(\overline{R})$ of length n , where $\overline{R} = R/\mathbf{x}R$ and $\overline{C}_i = \overline{R} \otimes_R C_i$ for $i = 0, 1, \dots, n$.

Proof. By [4, Theorem 5.1], \overline{C}_i is a semidualizing \overline{R} -module for all $i = 0, \dots, n$. Also, by [4, Theorem 5.10] and [17, Proposition 4.2.18], $[\overline{C}_i] \triangleleft [\overline{C}_{i-1}]$ for $i = 1, \dots, n$. Therefore $[\overline{C}_n] \triangleleft \dots \triangleleft [\overline{C}_1] \triangleleft [\overline{C}_0]$ is a chain in $\mathfrak{G}_0(\overline{R})$ of length n . Now we show that this chain is suitable.

For a semidualizing \overline{R} -module C , set $(-)^{\overline{\dagger}C} = \text{Hom}_{\overline{R}}(-, C)$. For each sequence of integers $\mathbf{i} = \{i_1, \dots, i_j\}$ such that $j \geq 0$ and $1 \leq i_1 < \dots < i_j \leq n$, set $\mathbf{C}_{\mathbf{i}} = \overline{C}_0^{\overline{\dagger}_{\overline{C}_{i_1}} \overline{\dagger}_{\overline{C}_{i_2}} \dots \overline{\dagger}_{\overline{C}_{i_j}}}$. (When $j = 0$, set $\mathbf{C}_{\mathbf{i}} = \mathbf{C}_{\emptyset} = \overline{C}_0 = \overline{R}$).

By induction on j we prove that

$$\mathbf{C}_{\mathbf{i}} = \overline{C}_0^{\overline{\dagger}_{\overline{C}_{i_1}} \overline{\dagger}_{\overline{C}_{i_2}} \dots \overline{\dagger}_{\overline{C}_{i_j}}} \cong \overline{C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_j}}}} = \overline{R} \otimes_R C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_j}}} = \overline{C}_{\mathbf{i}}$$

and $\mathbf{C}_{\mathbf{i}}$ is totally \overline{C}_t -reflexive for all \mathbf{i} and t with $i_j \leq t \leq n$.

When $j = 0$, there is nothing to prove. If $j = 1$, then $\mathbf{C}_{\mathbf{i}} = \text{Hom}_{\overline{R}}(\overline{C}_0, \overline{C}_{i_1}) \cong \overline{C_0^{\dagger_{C_{i_1}}}}$. As $[C_n] \triangleleft \dots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain, $C_0^{\dagger_{C_{i_1}}}$ is totally C_t -reflexive for all $i_1 \leq t \leq n$. Hence $\mathbf{C}_{\mathbf{i}}$ is totally \overline{C}_t -reflexive for all $i_1 \leq t \leq n$, by [4, Theorem 5.10].

Let $j > 1$. We have $\mathbf{C}_{\mathbf{i}} = (\overline{C}_0^{\overline{\dagger}_{\overline{C}_{i_1}} \overline{\dagger}_{\overline{C}_{i_2}} \dots \overline{\dagger}_{\overline{C}_{i_{j-1}}}})^{\overline{\dagger}_{\overline{C}_{i_j}}}$ and by induction

$$\overline{C}_0^{\overline{\dagger}_{\overline{C}_{i_1}} \overline{\dagger}_{\overline{C}_{i_2}} \dots \overline{\dagger}_{\overline{C}_{i_{j-1}}}} \cong \overline{C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_{j-1}}}}}$$

is totally \overline{C}_t -reflexive for all $i_{j-1} \leq t \leq n$. As $C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_{j-1}}}}$ is totally C_{i_j} -reflexive, we obtain the isomorphism

$$\overline{\text{Hom}_{\overline{R}}(C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_{j-1}}}}, \overline{C}_{i_j})} \cong \overline{\text{Hom}_R(C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_{j-1}}}}, C_{i_j})}.$$

Therefore $\mathbf{C}_{\mathbf{i}} \cong \overline{C_0^{\dagger_{C_{i_1}} \dagger_{C_{i_2}} \dots \dagger_{C_{i_j}}}} = \overline{C}_{\mathbf{i}}$ and $\mathbf{C}_{\mathbf{i}}$ is totally \overline{C}_t -reflexive for all $i_j \leq t \leq n$, by [4, Theorem 5.10]. Thus $[\overline{C}_n] \triangleleft \dots \triangleleft [\overline{C}_1] \triangleleft [\overline{C}_0]$ is a suitable chain in $\mathfrak{G}_0(\overline{R})$. \square

For the remaining part of this paper we assume that (R, \mathfrak{m}, k) is an Artinian local ring and that all modules are finite.

Definition 3.2. [8, Definitions 4.1 and 4.2] The modules K_1, K_2, \dots, K_n are said to be weakly Tor-independent if $\text{amp}(\otimes_{1 \leq i \leq n}^L K_i) = 0$. These modules are said to be strongly Tor-independent if for any subset $\Lambda \subseteq [n]$ we have $\text{amp}(\otimes_{i \in \Lambda}^L K_i) = 0$.

In the case $n = 2$ both notions are equivalent to the classical Tor-independence, i.e., to the condition that $\text{Tor}_{>0}^R(K_1, K_2) = 0$.

Theorem 3.3. [8, Theorem 4.5] *If the modules K_1, K_2, \dots, K_n are non-free and strongly Tor-independent, then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Poincaré series of k has the form $1/\prod_{i=1}^n (1 - d_i t)$ for some positive integers d_i .*

Conjecture 3.4. [8, Conjecture 4.6] If $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a chain in $\mathfrak{G}_0(R)$ of length n , then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Poincaré series of k has the form $1/\prod_{i=1}^n (1 - d_i t)$ for some positive integers d_i .

In the next result we prove the conjecture for suitable chains.

Theorem 3.5. *Let $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ be a suitable chain in $\mathfrak{G}_0(R)$ of length n , then $\mathfrak{m}^n \neq 0$. If, under the same conditions, $\mathfrak{m}^{n+1} = 0$, then the Poincaré series of k has the form $1/\prod_{i=1}^n (1 - d_i t)$ for some positive integers d_i .*

Proof. The non-free semidualizing modules B_1, B_2, \dots, B_n are strongly Tor-independent, where B_i is as in Notation 2.3, for each $i \in [n]$. Indeed, by Proposition 2.6, for each sequence of integers $\mathbf{i} = \{i_1, \dots, i_j\}$ with $j \geq 1$ and $1 \leq i_1 < \cdots < i_j \leq n$, $B_{\mathbf{i}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}$ is a semidualizing R -module. Thus $\text{amp}(B_{i_1} \otimes_R^L \cdots \otimes_R^L B_{i_j}) = \text{amp}(B_{\mathbf{i}}) = 0$. Therefore the assertion concludes by Theorem 3.3. \square

Note that Theorem 3.5 implies [8, Theorem 4.8]. Indeed, for $n \leq 3$ any chain of the form $[D] = [C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0] = [R]$, where D is dualizing, is also a suitable chain of length n . At this moment we do not know whether a chain for which Conjecture 3.4 holds true, is a suitable chain. However we guess that, if $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a chain in $\mathfrak{G}_0(R)$ such that the semidualizing modules B_1, \dots, B_n are strongly Tor-independent, then this chain is suitable.

Remark 3.6. When (S, \mathfrak{n}) is a Cohen-Macaulay local ring with dimension d and $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(S)$ of length n , then, by Proposition 3.1, $[\overline{C}_n] \triangleleft \cdots \triangleleft [\overline{C}_1] \triangleleft [\overline{C}_0]$ is also a suitable chain in $\mathfrak{G}_0(\overline{S})$ of length n , where $\overline{S} = S/\mathbf{x}S$, $\overline{C}_i = \overline{S} \otimes_S C_i$ and $\mathbf{x} = x_1, \dots, x_d$ is an S -regular sequence. If $\overline{\mathfrak{m}}^{n+1} = 0$, then by Theorem 3.5, the Poincaré series of $S/\mathfrak{n} \cong \overline{S}/\overline{\mathfrak{n}}$ has the form $1/\prod_{i=1}^n (1 - d_i t)$ for some positive integers d_i .

Also, one may see that the set $\mathfrak{G}_0(S)$ can not contain suitable chains of arbitrary length. Indeed, the length of a suitable chain in $\mathfrak{G}_0(S)$ is less than the number $l = \min \{i > 0 \mid \overline{\mathfrak{m}}^i = 0\}$. In [15], some other upper bounds for the length of chains of semidualizing modules are given.

Definition 3.7. [8, Definition 4.9] An Artinian ring R is called $SD(n)$ -full if the following conditions are satisfied.

- (i) $\mathfrak{m}^{n+1} = 0$.
- (ii) There are strongly Tor-independent non-free semidualizing modules K_1, K_2, \dots, K_n such that for any subset $\Lambda \subseteq [n]$ the module $\otimes_{i \in \Lambda} K_i$ is semidualizing.

Theorem 3.8. *An Artinian ring R is $SD(n)$ -full if and only if there is a suitable chain in $\mathfrak{G}_0(R)$ of length n and $\mathfrak{m}^{n+1} = 0$.*

Proof. First we assume that $\mathfrak{m}^{n+1} = 0$ and $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n . Then, by Proposition 2.6, the non-free semidualizing modules B_1, B_2, \dots, B_n are strongly Tor-independent and, for each subset $\mathbf{i} = \{i_1, \dots, i_j\} \subseteq [n]$, $B_{\mathbf{i}} = B_{i_1} \otimes_R \cdots \otimes_R B_{i_j}$ is a semidualizing R -module. Therefore R is $SD(n)$ -full.

For the converse, assume that K_1, K_2, \dots, K_n are strongly Tor-independent non-free semidualizing modules such that for any subset $\Lambda \subseteq [n]$ the module $\otimes_{i \in \Lambda} K_i$ is semidualizing. Set $C_0 = R$ and $C_j = \otimes_{1 \leq i \leq j} K_i$ for each $j \in [n]$. By [8, Proposition 3.5], it can be seen that the semidualizing modules C_0, C_1, \dots, C_n form a chain in $\mathfrak{G}_0(R)$ and then the sequence

$$(3.1) \quad [C_n] \triangleleft [C_{n-1}] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$$

is a chain in $\mathfrak{G}_0(R)$ of length n . We show that (3.1) is a suitable chain. For each sequence $\mathbf{i} = \{i_1, \dots, i_j\} \subseteq [n]$ such that $j \geq 0$ and $1 \leq i_1 < \cdots < i_j \leq n$, we have $C_{\mathbf{i}} = C_0^{\dagger_{C_{i_1}}} \cdots C_{i_j}^{\dagger}$. (When $j = 0$, set $C_{\mathbf{i}} = C_{\emptyset} = C_0$). Note that for each t , $i_j \leq t \leq n$,

$$C_t = \otimes_{1 \leq l \leq t} K_l = (\otimes_{1 \leq l \leq i_1} K_l) \otimes_R (\otimes_{i_1 < l \leq i_2} K_l) \otimes_R \cdots \otimes_R (\otimes_{i_{j-1} < l \leq i_j} K_l) \otimes_R (\otimes_{i_j < l \leq t} K_l).$$

On the other hand, by [8, Proposition 3.5], we have

$$C_0^{\dagger_{C_{i_1}}} \cdots C_{i_2}^{\dagger} = \text{Hom}_R(\otimes_{1 \leq l \leq i_1} K_l, \otimes_{1 \leq l \leq i_2} K_l) \cong \otimes_{i_1 < l \leq i_2} K_l$$

and so

$$C_0^{\dagger_{C_{i_1}}} \cdots C_{i_2}^{\dagger} \cdots C_{i_3}^{\dagger} = \text{Hom}_R(\otimes_{i_1 < l \leq i_2} K_l, \otimes_{1 \leq l \leq i_3} K_l) \cong (\otimes_{1 \leq l \leq i_1} K_l) \otimes_R (\otimes_{i_2 < l \leq i_3} K_l).$$

By proceeding in this way and using [8, Proposition 3.5], one may see that $C_{\mathbf{i}}$ is totally C_t -reflexive. Thus the sequence (3.1) is a suitable chain. \square

Example 3.9. A ring with $\mathfrak{m}^{n+1} = 0$ is $SD(n)$ -full if and only if there is a suitable chain in $\mathfrak{G}_0(R)$ of length n .

Let F be a field. Set $S_i = F \ltimes F^{a_i}$ for all $1 \leq i \leq n$, where $a_i > 1$. Then S_i is an Artinian ring with dualizing module $D_i = \text{Hom}_F(S_i, F)$. As type $S_i = a_i \neq 1$, the ring S_i is not Gorenstein. Set $S = \otimes_F^{1 \leq i \leq n} S_i$. By [8, Example 4.11], S is $SD(n)$ -full. Now, we construct a suitable chain of length n in $\mathfrak{G}_0(S)$. If K_i and M_i are S_i -modules such that K_i is finite, then there is S -module isomorphism $\text{Hom}_S(\otimes_F^{1 \leq i \leq n} K_i, \otimes_F^{1 \leq i \leq n} M_i) \cong \otimes_F^{1 \leq i \leq n} \text{Hom}_{S_i}(K_i, M_i)$, by [17, Proposition A.1.5]. When K_i is semidualizing, by [17, Proposition 2.3.6], $\otimes_F^{1 \leq i \leq n} K_i$ is also a semidualizing S -module, and $\otimes_F^{1 \leq i \leq n} M_i$ is totally $\otimes_F^{1 \leq i \leq n} K_i$ -reflexive by [17, Proposition 5.3.3], whenever M_i is totally K_i -reflexive. Note that if we take, for each i , a module $K_i \in \{S_i, D_i\}$, then the module $\otimes_F^{1 \leq i \leq n} K_i$ is semidualizing S -module. Set $C_0 = S$ and $C_j = (\otimes_F^{1 \leq i \leq j} D_i) \otimes_F (\otimes_F^{j < i \leq n} S_i)$ for all $1 \leq j \leq n$. Therefore we obtain the

chain $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ in $\mathfrak{G}_0(S)$ of length n . We show that this chain is suitable. Assume that $\mathbf{u} = \{u_1, \dots, u_j\}$ is a sequence of integers such that $1 \leq u_1 < \cdots < u_j \leq n$ and $C_{\mathbf{u}} = C_0^{\dagger_{C_{u_1}}} \cdots \dagger_{C_{u_j}}$. We have $C_0^{\dagger_{C_{u_1}}} \cong C_{u_1}$,

$$\begin{aligned} C_0^{\dagger_{C_{u_1}}} \dagger_{C_{u_2}} &\cong \text{Hom}_S(C_{u_1}, C_{u_2}) \cong (\otimes_F^{1 \leq i \leq u_1} S_i) \otimes_F (\otimes_F^{u_1 < i \leq u_2} D_i) \otimes_F (\otimes_F^{u_2 < i \leq n} S_i), \text{ and} \\ C_0^{\dagger_{C_{u_1}}} \dagger_{C_{u_2}} \dagger_{C_{u_3}} &= \text{Hom}_S(C_0^{\dagger_{C_{u_1}}} \dagger_{C_{u_2}}, C_{u_3}) \\ &\cong (\otimes_F^{1 \leq i \leq u_1} D_i) \otimes_F (\otimes_F^{u_1 < i \leq u_2} S_i) \otimes_F (\otimes_F^{u_2 < i \leq u_3} D_i) \otimes_F (\otimes_F^{u_3 < i \leq n} S_i). \end{aligned}$$

By proceeding in this way one obtains the following isomorphism

$$C_{\mathbf{u}} \cong \begin{cases} (\otimes_F^{1 \leq i \leq u_1} S_i) \otimes_F (\otimes_F^{u_1 < i \leq u_2} D_i) \otimes_F \cdots \otimes_F (\otimes_F^{u_{j-1} < i \leq u_j} D_i) \otimes_F (\otimes_F^{u_j < i \leq n} S_i) & \text{if } j \text{ is even,} \\ (\otimes_F^{1 \leq i \leq u_1} D_i) \otimes_F (\otimes_F^{u_1 < i \leq u_2} S_i) \otimes_F \cdots \otimes_F (\otimes_F^{u_{j-1} < i \leq u_j} D_i) \otimes_F (\otimes_F^{u_j < i \leq n} S_i) & \text{if } j \text{ is odd.} \end{cases}$$

As $C_t = (\otimes_F^{1 \leq i \leq t} D_i) \otimes_F (\otimes_F^{t < i \leq n} S_i)$, one can see that $C_{\mathbf{u}}$ is totally C_t -reflexive for all $u_j \leq t \leq n$. Hence $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(S)$.

Note that $B_j = \text{Hom}_S(C_{j-1}, C_j) \cong S_1 \otimes_F \cdots \otimes_F S_{j-1} \otimes_F D_j \otimes_F S_{j+1} \otimes_F \cdots \otimes_F S_n$ for all $1 \leq j \leq n$. The semidualizing modules B_1, \dots, B_n are strongly Tor-independent and for any subset $\Lambda \subseteq [n]$ the module $\otimes_{i \in \Lambda} B_i$ is semidualizing.

The next result follows from [8, Proposition 5.4] and the proof of Theorem 3.8.

Lemma 3.10. *Let R be an Artinian ring with $\mathfrak{m}^{n+1} = 0$. Assume that $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n . Then for each $i \in [n]$ the Poincaré series of B_i is $P_{B_i}^R(t) = (\beta_0^R(B_i) - t)/(1 - \beta_0^R(B_i)t)$ and the Bass series of $B_{[n] \setminus i}$ is $I_R^{B_{[n] \setminus i}}(t) = (\mu_R^0(B_{[n] \setminus i}) - t)/(1 - \mu_R^0(B_{[n] \setminus i})t)$.*

Proposition 3.11. *Let R be an Artinian ring with $\mathfrak{m}^{n+1} = 0$. Assume that $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(R)$ of length n . Then for each $i \in [n]$ the Poincaré series of C_i is*

$$P_{C_i}^R(t) = \frac{\prod_{j=1}^i (\beta_0^R(B_j) - t)}{\prod_{j=1}^i (1 - \beta_0^R(B_j)t)},$$

$I_R^{C_n}(t) = 1$, and for $i \neq n$, the Bass series of C_i is

$$I_R^{C_i}(t) = \frac{\prod_{j=i+1}^n (\beta_0^R(B_j) - t)}{\prod_{j=i+1}^n (1 - \beta_0^R(B_j)t)} = \frac{\prod_{j=i+1}^n (\mu_R^0(B_{[n] \setminus j}) - t)}{\prod_{j=i+1}^n (1 - \mu_R^0(B_{[n] \setminus j})t)}.$$

Proof. As $\mathfrak{m}^{n+1} = 0$, this suitable chain is of maximum length, by Theorem 3.5. Thus the module C_n is the dualizing R -module. Hence the Bass series of C_n is $I_R^{C_n}(t) = 1$. For $i \in [n]$, we have $C_i \cong B_1 \otimes_R \cdots \otimes_R B_i$, by Fact 2.4. Therefore $P_{C_i}^R(t) = P_{B_1}^R(t) P_{B_2}^R(t) \cdots P_{B_i}^R(t)$, by [2, Lemma 1.5.3]. Hence, by Lemma 3.10, one gets

$$P_{C_i}^R(t) = \left(\frac{\beta_0^R(B_1) - t}{1 - \beta_0^R(B_1)t} \right) \left(\frac{\beta_0^R(B_2) - t}{1 - \beta_0^R(B_2)t} \right) \cdots \left(\frac{\beta_0^R(B_i) - t}{1 - \beta_0^R(B_i)t} \right).$$

Also, we have $C_i \cong B_1 \otimes_R \cdots \otimes_R B_i \cong \text{Hom}_R(B_{\{i+1, \dots, n\}}, B_{[n]})$, by Proposition 2.6. Thus, by [2, Lemma 1.5.3], one gets

$$I_R^{C_i}(t) = P_{B_{\{i+1, \dots, n\}}}^R(t) I_R^{B_{[n]}}(t) = P_{B_{i+1}}^R(t) \cdots P_{B_n}^R(t) I_R^{B_{[n]}}(t).$$

As $B_{[n]} = B_1 \otimes_R \cdots \otimes_R B_n \cong C_n$, one has $I_R^{B_{[n]}}(t) = I_R^{C_n}(t) = 1$. Therefore, by Lemma 3.10,

$$I_R^{C_i}(t) = P_{B_{i+1}}^R(t) \cdots P_{B_n}^R(t) = \left(\frac{\beta_0^R(B_{i+1}) - t}{1 - \beta_0^R(B_{i+1})t} \right) \cdots \left(\frac{\beta_0^R(B_n) - t}{1 - \beta_0^R(B_n)t} \right).$$

By Proposition 2.6, $B_{[n] \setminus j} \cong \text{Hom}_R(B_j, B_{[n]})$ and so $I_R^{B_{[n] \setminus j}}(t) = P_{B_j}^R(t) I_R^{B_{[n]}}(t) = P_{B_j}^R(t)$. Note that $\mu^0(B_{[n] \setminus j}) = \beta_0(B_j)$ and then the final equality follows. \square

From Proposition 3.11, the Bass series of R is

$$I_R(t) = I_R^{\text{Hom}_R(C_1, C_1)}(t) = P_{C_1}^R(t) I_R^{C_1}(t) = I_R^{B_{[n] \setminus 1}}(t) I_R^{C_1}(t) = \frac{\prod_{j=1}^n (\mu_R^0(B_{[n] \setminus j}) - t)}{\prod_{j=1}^n (1 - \mu_R^0(B_{[n] \setminus j})t)}, \text{ and}$$

$$I_R(t) = P_{C_n}^R(t) = \frac{\prod_{j=1}^n (\beta_0^R(B_j) - t)}{\prod_{j=1}^n (1 - \beta_0^R(B_j)t)}.$$

For each $i \in [n]$, $I_R^{B_{[n] \setminus i}}(t) = \alpha_i + (\alpha_i^2 - 1)t + \alpha_i(\alpha_i^2 - 1)t^2 + \alpha_i^2(\alpha_i^2 - 1)t^3 + \cdots$, where $\alpha_i = \mu_R^0(B_{[n] \setminus i})$ (see [8, Propositions 5.1, 5.2 and 5.4]). Thus $\{\mu_R^j(B_{[n] \setminus i})\}$ is strictly increasing (indeed, it has exponential growth, since $\mu_R^j(B_{[n] \setminus i}) \geq \alpha_i^j$ for all $j \geq 1$). Therefore $\{\mu_R^j(R)\}$ is also strictly increasing whenever $n \geq 1$, since $I_R(t) = I_R^{B_{[n] \setminus 1}}(t) I_R^{B_{[n] \setminus 2}}(t) \cdots I_R^{B_{[n] \setminus n}}(t)$. Also, similar to the proof of [15, Theorem 3.5], it is easy to see that $\{\mu_R^j(R)\}$ is bounded below by a polynomial in j of degree $n - 1$.

Remark 3.12. Assume that (S, \mathfrak{n}) is a Cohen-Macaulay local ring with dimension d and $\mathbf{x} = x_1, \dots, x_d$ is an S -regular sequence. Set $\overline{S} = S/\mathbf{x}S$. As $\text{Ext}_S^i(S/\mathfrak{n}, S) \cong \text{Ext}_{\overline{S}}^{i-d}(S/\mathfrak{n}, \overline{S})$ for all $i \geq d$, we have $\mu_S^i(S) = \mu_{\overline{S}}^{i-d}(\overline{S})$, for all $i \geq d$. Thus we get $I_S(t) = t^d I_{\overline{S}}(t)$. Now, if $[C_n] \triangleleft \cdots \triangleleft [C_1] \triangleleft [C_0]$ is a suitable chain in $\mathfrak{G}_0(S)$ of length n and $\overline{\mathfrak{n}}^{n+1} = 0$, then, by Proposition 3.1 and the above discussion, $I_S(t)$ has a very specific form and $\{\mu_S^i(S)\}$ is strictly increasing.

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